Products of random matrices

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We derive analytic expressions for infinite products of random 2×2 matrices. The determinant of the target matrix is log-normally distributed, whereas the remainder is a surprisingly complicated function of a parameter characterizing the norm of the matrix and a parameter characterizing its skewness. The distribution may have importance as an uncommitted prior in statistical image analysis.

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I. INTRODUCTION

Considerable effort has been invested over the past half century in determining the spectral properties of ensembles of matrices with randomly chosen elements and in discovering the remarkably broad applicability of these results to systems of physical interest. In spite of a similarly rich set of potential applications (e.g., in the statistical theory of Markov processes and in various chaotic dynamical systems in classical physics), the properties of products of random matrices have received considerably less attention. See Ref. [1] for a survey of products of random matrices in statistics and Ref. [2] for a review of physics applications.

The purpose of the present manuscript is to consider in some detail the limit for $N\to\infty$ of the ensemble of matrices

$$Y=\left(1+\sqrt{\frac{\tau}{N}}X_1\right)\times\left(1+\sqrt{\frac{\tau}{N}}X_2\right)\cdots\left(1+\sqrt{\frac{\tau}{N}}X_N\right), \tag{1}$$

where $\tau>0$ is a real parameter and the X_n are real $d\times d$ matrices with all elements drawn at random on a distribution of zero mean and unit variance. If this distribution has compact support, the probability that the matrix Y should become nonpositive definite vanishes for $N\to\infty$. In one dimension, $d=1$, it is well known from the law of large numbers that $\log Y$ has a Gaussian distribution, but because of the noncommutativity of matrix products, the distribution is much more complicated for $d\geqslant 2$.

In this paper we derive some general properties for the limiting distribution $\mathcal{P}(Y)$ and determine it explicitly for $d=2$. In Sec. II we establish a compact diffusion equation for the distribution valid for any d. In Sec. III we derive a simple expression for any average over the distribution, and we show that the determinant $\det[Y]$ has a log-normal distribution. Secs. IV and V will be devoted to the determination of the explicit form of $\mathcal{P}$ for $d=2$. We shall first write the diffusion equation using an appropriate parametrization of Y. The resulting partial differential equation will then be solved subject to the boundary condition that $\mathcal{P}(Y)$ supports only the identity matrix in the limit of $\tau\to 0$. This explicit solution will require new integrals involving Jacobi functions. The derivation of these integrals will be given in the Appendix.

II. THE DIFFUSION EQUATION

The normalized probability distribution is (for given N and variable τ)

$$\mathcal{P}_N(Y,\tau)=\left\langle \delta\left[Y-\prod_{n=1}^{N}\left(1+\sqrt{\frac{\tau}{N}}X_n\right)\right]\right\rangle_{X_1,\ldots,X_N}, \tag{2}$$

where the integrand is a product of δ functions for each matrix element of Y and the average runs over all the random matrices. Pealing off the Nth factor in the product and using only that the X_n are statistically independent, we derive the following exact recursion relation

$$\mathcal{P}_N(Y,\tau)=\left\langle \det\left[1+\sqrt{\frac{\tau}{N}}X\right]^{-d}\times\mathcal{P}_{N-1}\left[Y\left(1+\sqrt{\frac{\tau}{N}}X\right)^{-1},\tau\frac{N-1}{N}\right]\right\rangle_X, \tag{3}$$

where the average is over the Nth random matrix, here renamed X. The determinantal prefactor of $\mathcal{P}_{N-1}$ is the Jacobi determinant arising from the general matrix rule

$$\delta[Y-ZM]=\frac{\delta[YM^{-1}-Z]}{\det\left[\frac{\partial(ZM)}{\partial Z}\right]}, \tag{4}$$

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with $M=1+\sqrt{\tau/N}\,X$. Since

$$\frac{\partial(ZM)_{ij}}{\partial Z_{k\ell}}=\delta_{ik}M_{\ell j}, \tag{5}$$

the Jacobian is block-diagonal with d identical blocks, and the prefactor follows.

The recursion relation (3) is of the Markovian type with the initial distribution $\mathcal{P}_0(Y,\tau)=\delta[Y-1]$. It converges for $N\to\infty$ under very general conditions (which we shall not discuss here) towards a limiting distribution $\mathcal{P}(y,\tau)=\lim_{N\to\infty}\mathcal{P}_N(y,\tau)$. Expanding the recursion relation to $\mathcal{O}(1/N)$ and using the fact that all the matrix elements of X are statistically independent with zero mean and unit variance,

$$\langle X_{ij}\rangle_X=0, \quad \langle X_{ij}X_{kl}\rangle_X=\delta_{ik}\delta_{jl}, \tag{6}$$

we obtain to leading order

$$\mathcal{P}_N=\mathcal{P}_{N-1}+\frac{\tau}{N}\left(-\frac{\partial\mathcal{P}_{N-1}}{\partial\tau}+\frac{1}{2}d^2(d+1)\mathcal{P}_{N-1}+(d+1)Y_{ij}\frac{\partial\mathcal{P}_{N-1}}{\partial Y_{ij}}+\frac{1}{2}Y_{ik}Y_{jk}\frac{\partial^2\mathcal{P}_{N-1}}{\partial Y_{i\ell}\partial Y_{j\ell}}\right),$$

with implicit summation over all repeated indices. The assumed convergence towards a limiting distribution requires the expression in the parentheses to vanish in the limit, so that

$$\frac{\partial\mathcal{P}}{\partial\tau}=\frac{1}{2}d^2(d+1)\mathcal{P}+(d+1)Y_{ij}\frac{\partial\mathcal{P}}{\partial Y_{ij}}+\frac{1}{2}Y_{ik}Y_{jk}\frac{\partial^2\mathcal{P}}{\partial Y_{i\ell}\partial Y_{j\ell}}. \tag{7}$$

This is a diffusion equation of the Fokker-Planck type with τ playing the role of time. It must be solved subject to the initial condition that $\mathcal{P}(y,0)=\delta[Y-1]$.

Both the diffusion equation and the initial condition are invariant with respect to an orthogonal transformation $Y\to M^\top YM$, where M is an orthogonal matrix satisfying $M^\top M=1$. Since the number of free parameters in an orthogonal transformation is $\frac{1}{2}d(d-1)$, the number of "dynamic" variables in the distribution is $d^2-\frac{1}{2}d(d-1)=\frac{1}{2}(d+1)$. Since the distribution only has support for $\det[Y]>0$, this number consists of d independent eigenvalues and $\frac{1}{2}d(d-1)$ rotation angles in a singular value decomposition.

For $d=1$ the solution to Eq. (7) which approaches $\delta[Y-1]$ for $\tau\to 0$ is

$$\mathcal{P}_{d=1}(Y)=\frac{1}{Y\sqrt{2\pi\tau}}\exp\left[-\frac{(\log Y+\tau/2)^2}{2\tau}\right]. \tag{8}$$

As expected, it is a log-normal distribution.

III. AVERAGES

Remarkably, Eq. (7) may be written in the much simpler form

$$\frac{\partial\mathcal{P}}{\partial\tau}=\frac{1}{2}\frac{\partial^2(Y_{ik}Y_{jk}\mathcal{P})}{\partial Y_{i\ell}\partial Y_{j\ell}} \tag{9}$$

without any explicit reference to d. Defining the average of a function $f(Y)$ by

$$\langle f\rangle=\int f(Y)\mathcal{P}(Y)\,dY \tag{10}$$

with $dY=\Pi_{ij}dY_{ij}$, we obtain from Eq. (9)

$$\frac{\partial\langle f\rangle}{\partial\tau}=\frac{1}{2}\left\langle Y_{ik}Y_{jk}\frac{\partial^2 f}{\partial Y_{i\ell}\partial Y_{j\ell}}\right\rangle. \tag{11}$$

This equation permits in principle the determination of the moment of any product of matrix elements. The first two are found to be

$$\langle Y_{ij}\rangle=\delta_{ij}, \tag{12}$$

$$\langle Y_{ij}Y_{kl}\rangle=e^{\tau d}\delta_{ik}\delta_{jl}. \tag{13}$$

The exponential growth of the averages with "time" τ is a consequence of the multiplicative nature of the problem.

The determinant $D=\det[Y]$ is, according to the definition of the product (1), an infinite product of random real numbers that converge towards unity, and $\log D$ must have a Gaussian distribution according to the law of large numbers. Its mean and variance are, however, different from those of the one-dimensional distribution (8). The distribution of the determinant is also an average

$$F(D)=\langle\delta(D-\det[Y])\rangle. \tag{14}$$

Using the fact that

$$\frac{\partial\det[Y]}{\partial Y_{ij}}=\det[Y]Y^{-1}_{ji}, \tag{15}$$

we obtain the following equation for F:

$$\frac{\partial F}{\partial\tau}=\frac{1}{2}d\frac{\partial^2(D^2F)}{\partial D^2}=d\left(F+2D\frac{\partial F}{\partial D}+\frac{1}{2}D^2\frac{\partial^2F}{\partial D^2}\right). \tag{16}$$

Apart from the factor d in front, this is identical to the diffusion equation (9) in one dimension. Consequently the determinant has a log-normal distribution

$$F(D)=\frac{1}{D\sqrt{2\pi\tau d}}\exp\left[-\frac{(\log D+\tau d/2)^2}{2\tau d}\right], \tag{17}$$

which is obtained from Eq. (8) by replacing τ by τd. The distribution has support only for positive values of D. It can be shown in general (and we shall demonstrate it explicitly for $d=2$ below) that the distribution of the determinant factorizes in $\mathcal{P}$.

IV. THE CASE $d=2$

The first nontrivial case is $d=2$ where the general matrix is first parametrized using a quaternion or 4-vector notation

$$Y=\begin{pmatrix} Y_0+Y_3 & Y_1-Y_2 \\ Y_1+Y_2 & Y_0-Y_3 \end{pmatrix}. \tag{18}$$

In this representation the determinant becomes a metric with two "space" and two "time" dimensions

$$D=Y_0^2-Y_1^2+Y_2^2-Y_3^2. \tag{19}$$

The structure of this expression and the positivity of D suggest the following parametrization in terms of one imaginary and two real angles:

$$Y_0=\sqrt{D}\cosh\psi\cos\theta, \tag{20a}$$

$$Y_1=\sqrt{D}\sinh\psi\cos\phi, \tag{20b}$$

$$Y_2=\sqrt{D}\cosh\psi\sin\theta, \tag{20c}$$

$$Y_3=\sqrt{D}\sinh\psi\sin\phi. \tag{20d}$$

The Jacobi determinant of the transformation from $\{Y_0,Y_1,Y_2,Y_3\}$ to $\{D,\psi,\theta,\phi\}$ is simply

$$J\sim D\sinh\psi\cosh\psi. \tag{21}$$

Orthogonal 2×2 matrices are generated by the matrix $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, which is associated with Y_2. Thus, an orthogonal transformation rotates the angle ϕ, and $\mathcal{P}(Y,\tau)$ must be independent of ϕ as indicated above.

In these variables the diffusion equation (7) simplifies to

$$\frac{\partial\mathcal{P}}{\partial\tau}=6\mathcal{P}+6D\frac{\partial\mathcal{P}}{\partial D}+D^2\frac{\partial^2\mathcal{P}}{\partial D^2}+\frac{1}{4}(1+\tanh^2\psi)\frac{\partial^2\mathcal{P}}{\partial\theta^2}+\frac{1}{4}(\tanh\psi+\coth\psi)\frac{\partial\mathcal{P}}{\partial\psi}+\frac{1}{4}\frac{\partial^2\mathcal{P}}{\partial\psi^2}. \tag{22}$$

Taking into account the factor of D in the Jacobi determinant, we replace the original distribution $\mathcal{P}$ with the product of the determinant distribution $F(D)$ given in Eq. (17) and an as yet unknown function of ψ and θ,

$$\mathcal{P}=\frac{1}{D}F(D)G(\psi,\theta), \tag{23}$$

and find that G satisfies the diffusion equation

$$\frac{\partial G}{\partial\tau}=\frac{1}{4}(1+\tanh^2\psi)\frac{\partial^2 G}{\partial\theta^2}+\frac{1}{4}(\tanh\psi+\coth\psi)\frac{\partial G}{\partial\psi}+\frac{1}{4}\frac{\partial^2 G}{\partial\psi^2}. \tag{24}$$

The corresponding normalization integral is found from the Jacobi determinant,

$$\int_0^{2\pi}d\theta\int_0^\infty d\psi 2\sinh 2\psi G(\psi,\theta)=1. \tag{25}$$

This normalization integrals (25) suggest that it is more convenient to employ still another variable

$$z=\cosh 2\psi=\frac{Y_0^2+Y_1^2+Y_2^2+Y_3^2}{Y_0^2-Y_1^2+Y_2^2-Y_3^2}. \tag{26}$$

With this variable the normalization integral takes the form

$$\int_0^{2\pi}d\theta\int_1^\infty dz G(z,\theta)=1, \tag{27}$$

and the diffusion equation (24) becomes

$$\frac{\partial G}{\partial\tau}=\frac{1}{4}\frac{2z}{z+1}\frac{\partial^2 G}{\partial\theta^2}+2z\frac{\partial G}{\partial z}+(z^2-1)\frac{\partial^2 G}{\partial z^2}. \tag{28}$$

This equation must be solved with the boundary condition that $\mathcal{P}(Y)$ in the limit $\tau\to 0$ reduces to a product of delta functions which select only the identity matrix. This evidently requires $Y_0\to 1$ and $Y_{1,2,3}\to 0$ and, consequently, $D\to 1$, $z\to 1$, and $\theta\to 0$. Since $F(D)\to\delta(D-1)$, the initial condition takes the form

$$G(z,\theta)\to\delta(z-1)\delta(\theta)\quad(\tau\to 0). \tag{29}$$

The limiting distribution should be approached from above (i.e., from $z>1$).

The form of the diffusion equation (28) reveals that G may naturally be expanded in a Fourier series

$$G(z,\theta)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}G_n(z)e^{in\theta} \tag{30}$$

with coefficients that obey

$$\frac{\partial G_n}{\partial\tau}=-\frac{1}{4}n^2\frac{2z}{z+1}G_n+2z\frac{\partial G_n}{\partial z}+(z^2-1)\frac{\partial^2 G_n}{\partial z^2}. \tag{31}$$

For the special case $n=0$, we recognize Legendre's differential operator on the right. The normalization condition only affects G_0 and becomes

$$\int_0^\infty dz\, G_0(z)=1. \tag{32}$$

The initial condition (29) implies that

$$G_n(z)\to\delta(z-1)\quad(\tau\to 0) \tag{33}$$

for all n.

V. EXPLICIT SOLUTION

All that remains is to determine the angular functions $G_n(z)$. One relatively simple way is to use Sturm-Liouville theory, and we now outline the main steps in this procedure.

The differential operator ("Hamiltonian") appearing on the right-hand side of Eq. (31) may be written

$$\mathcal{H}=\frac{\partial}{\partial z}(z^2-1)\frac{\partial}{\partial z}-\frac{n^2}{4}\frac{2z}{z+1}, \tag{34}$$

which shows that it is Hermitian. Let the spectral variable (which denumerates the eigenvalues and may be both discrete and continuous) be denoted r, and let $g_n^{(r)}(z)$ be the eigenfunction corresponding to the eigenvalue $\lambda_n^{(r)}$,

$$\mathcal{H}g_n^{(r)}(z)=\lambda_n^{(r)}g_n^{(r)}(z). \tag{35}$$

The Hermiticity of $\mathcal{H}$ guarantees that the eigenvalues are real and that the eigenfunctions are both orthogonal and complete on the interval $1\leqslant z<\infty$,

$$\int_1^\infty dz g_n^{(r)}(z)g_n^{(r')}(z)=\frac{\delta_{r,r'}}{\mu_n^{(r)}}, \tag{36}$$

$$\sum_r \mu_n^{(r)}g_n^{(r)}(z)g_n^{(r)}(z')=\delta(z-z'), \tag{37}$$

with a suitable measure, $\mu_n^{(r)}$.

The solution of the diffusion equation (31) with initial condition (33) takes the form

$$G_n(z,\tau)=\sum_r \mu_n^{(r)}g_n^{(r)}(1)g_n^{(r)}(z)\exp(\lambda_n^{(r)}\tau). \tag{38}$$

In view of the completeness (37), these functions indeed satisfy the initial conditions at $\tau=0$. The appearance of $g_n^{(r)}(1)$ in this expression requires the eigenfunctions to be regular at $z=1$.

We now present the complete solution of the eigenvalue problem. (Further details are given in the Appendix.) The eigenvalue spectrum contains discrete values (for $n\geqslant 2$) as well as a continuum

$$\lambda_n^{(r)}=\begin{cases} -\frac{1}{2}n^2-\frac{1}{4}+\left(\frac{n+1}{2}-k\right)^2, & k=1,2,\ldots,\lfloor\frac{n}{2}\rfloor,\\ -\frac{1}{2}n^2-\frac{1}{4}-t^2, & 0\leqslant t<\infty. \end{cases} \tag{39}$$

The properly normalized discrete eigenfunctions are Jacobi polynomials

$$g_n^{(k)}=\sqrt{\frac{n+1}{2}-k}\left(\frac{1+z}{2}\right)^{n/2}P_{-k}^{(0,n)}(z), \tag{40}$$

while the eigenfunctions in the continuum are Jacobi functions of complex index

$$g_n^{(t)}=\left(\frac{1+z}{2}\right)^{n/2}P_{-(n+1)/2+it}^{(0,n)}(z) \tag{41}$$

with the measure obtained from the integral (36) as

$$\mu_n^{(t)}=\begin{cases} t\tanh\pi t, & n \text{ even},\\ t\coth\pi t, & n \text{ odd}. \end{cases} \tag{42}$$

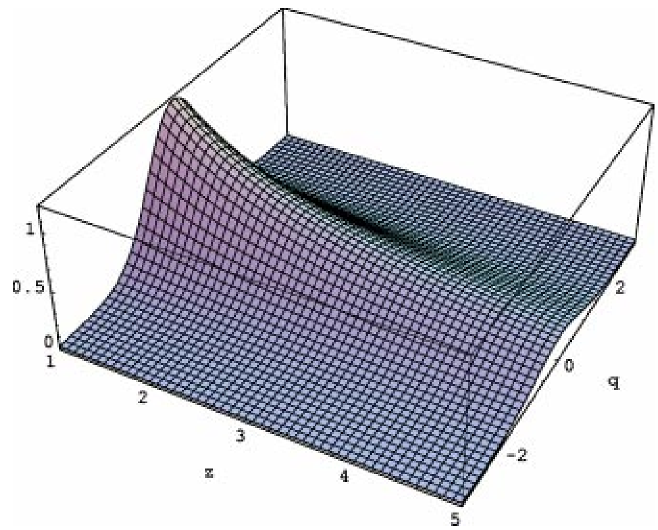

FIG. 1. Plot of $G(z,\theta)$ for $\tau=1$. Notice the characteristic log-normal tapering of the ridge as a function of z, and the nearly Gaussian distribution in θ around $\theta=0$.

The special case $n=0$ was stated without proof by Mehler in 1881 [3]. The general case is proven in the Appendix.

Since $g_n^{(t)}(1)=1$, the final solution becomes a simple superposition of the discrete and continuous contributions

$$G_n=G_n^{\text{disc}}+G_n^{\text{cont}} \tag{43}$$

where the discrete contribution (for $n\geqslant 2$) is

$$G_n^{\text{disc}}(z,\tau)=\left(\frac{1+z}{2}\right)^{n/2}\sum_{k=1}^{\lfloor n/2\rfloor}\left(\frac{n+1}{2}-k\right)\times P_{-k}^{(0,n)}(z)e^{-(n^2/2+1/4-[(n+1)/2-k]^2)\tau}. \tag{44}$$

The continuous contribution is

$$G_n^{\text{cont}}(z,\tau)=\left(\frac{1+z}{2}\right)^{n/2}\int_0^\infty dt\,\mu_n(t)\times P_{-(n+1)/2+it}^{(0,n)}(z)e^{-(n^2/2+1/4+t^2)\tau}, \tag{45}$$

with $\mu_n(t)$ given by Eq. (42). Thus, we arrive at the final result. The probability for drawing a given 2×2 matrix Y is

$$\mathcal{P}(Y,\tau)=\frac{F(D)}{2\pi D}\left(G_0(z,\tau)+2\sum_{n=1}^{\infty}G_n(z,\tau)\cos n\theta\right) \tag{46}$$

with $F(D)$ given by Eq. (17) and $G_n(z,\tau)$ given by Eqs. (43)–(45). As noted previously, the $G_n(z,\tau)$ are independent of the sign of n so that $\mathcal{P}$ is manifestly real. In Fig. 1 the function $G(z,\theta)$ (the expression in parenthesis) is plotted for $\tau=1$.

VI. NUMERIC APPROXIMATION

Given the relative complexity of our final analytic result, it is satisfying to note that it is easy to obtain a simple and accurate approximate form which is suitable for numerical applications. Specifically, consider Eq. (28). The complicated coupling between the variables θ and z are a consequence of the factor of $2z/(z+1)$ appearing in the first term on the right of this equation. This factor changes by only a factor of 2 over the interval $1 \leqslant z \leqslant \infty$. A simple separable approximation can thus be obtained by replacing this single factor by a constant, f. The most appropriate value of $1 \leqslant f \leqslant 2$ evidently depends on τ. With this replacement, our final result can be approximated as

$$\mathcal{P}(Y,\tau) \approx \frac{F(D)}{2\pi D} G_0(z,\tau) \sum_{n=-\infty}^{+\infty} \exp[in\theta - f\tau n^2/4]. \tag{47}$$

Here, the θ-dependent term is known as Fourier's ring. For $\tau=1$ and the choice $f=1.23$, this approximate form yields a root mean square error of 0.035. Comparison with the integral of the square of $\mathcal{P}$, this suggests an error of less than 2%. A maximum error of 4% is encountered at $z=0$ and $\theta=0$. Similar results are found for other values of τ using an appropiate value of f. The error increases slowly with increasing values of τ.

Additional approximations are possible and introduce little additional error. For example, Fourier's ring can be written in a form which converges rapidly for large τ:

$$\sum_{n=-\infty}^{+\infty} \exp[in\theta - f\tau n^2/4] = \sqrt{\frac{\pi}{f\tau}} \sum_{k=-\infty}^{+\infty} \exp[-(\theta-2k\pi)^2/4f\tau]. \tag{48}$$

We note also that $G_0(z,\tau)$ can be approximated with surprising accuracy as $A\exp[-\mu z^{\gamma}]$, where μ and γ are smooth functions of τ and A is a normalization constant.

VII. CONCLUSIONS

We have analytically derived the distribution of an infinite product of random 2×2 matrices. In statistical image analysis, it may be used as an uncommitted prior for morphing and warping [4], with desirable properties not shared by the usual priors based on elastic membranes. The distribution of such matrices may be evaluated numerically at a moderate cost in computer time and converges reasonably fast because of the strong exponential damping. We have also outlined a numeric approximation with sufficient precision for practical applications.

APPENDIX: JACOBI FUNCTIONS

The Jacobi functions are related to the hypergeometric functions

$$P^{(0,n)}_{-n/2-1/2+it}(z) = {}_2F_1\left(\frac{n+1}{2} - it, \frac{n+1}{2} + it; 1; \frac{1-z}{2}\right) \tag{A1}$$

with t real, and obey the orthogonality relation

$$\int_1^{\infty} \left(\frac{1+z}{2}\right)^n dz P^{(0,n)}_{-n/2-1/2+it}(z) P^{(0,n)}_{-n/2-1/2+it'}(z) = \frac{\delta(t-t')}{\mu_n(t)}. \tag{A2}$$

In order to find $\mu_n(t)$ for arbitrary n, it is helpful to consider the asymptotic form of these functions by using the standard relation for hypergeometric functions

$$\begin{aligned} F(a,b;c;z) = (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} & \\ \times F\left(a, c-b; a-b+1; \frac{1}{1-z}\right) & \\ + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} & \\ \times F\left(b, c-a; b-a+1; \frac{1}{1-z}\right). & \end{aligned} \tag{A3}$$

This form allows us to see that

$$P^{(0,n)}_{-n/2-1/2+it'}(z) \rightarrow 2|A(t)| z^{-n/2-1/2} \cos(\phi_t + t \ln z) \tag{A4}$$

as $z\rightarrow\infty$. Here,

$$A(t) = \frac{\Gamma(2it) 2^{n/2+1/2-it}}{\Gamma(n/2+1/2+it)\Gamma(-n/2+1/2+it)} \tag{A5}$$

and ϕ_t is the phase of $A(t)$. Using this asymptotic form, we can perform the integral in Eq. (A2) by using the variable $u=\log z$, adding a convergence factor of $\exp(-\mu u)$, and finally taking the limit $\mu\rightarrow 0$. The result is simply

$$|A(t)|^2 \left[\frac{2\mu}{\mu^2 + (t-t')^2}\right] 2^{-n}. \tag{A6}$$

The factor in brackets is a familiar representation of $2\pi\delta(t-t')$ in the limit $\mu\rightarrow 0$. Standard relations for the gamma function immediately yield Eq. (36). This confirms the results of Mehler [3] for the special case $n=0$. The extension to $n>0$ would appear to be new.

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